FREQUENCY CHARACTERISTICS OF A TEMPERATURE INTEGRATOR

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The frequency characteristics of a temperature integrator are determined. The calculations have been verified experimentally.

Thermal noise in a stream of a heat carrying fluid is suppressed by means of temperature integrators (TI). On account of the rather high thermal diffusivity of the material used in these devices, the heat content of the stream is distributed over the entire volume of such a device. As a result, the amplitude of thermal perturbations in the heat carrier becomes lower at the exit from than at the entrance to a temperature integrator.

The problem of noise suppression was considered in [1] with a temperature integrator of an infinitely high thermal diffusivity. A more rigorous analysis requires that processes in the space–time domain inside the temperature integrator also be taken into account.

The model of a temperature integrator selected for this analysis consists of a rectangular strip (length b, width l, thickness m) made of a heat conducting material and wetted on one side by the heat carrying fluid. The problem is solved under the following assumptions: a) the heat carrier mixes thoroughly across a transverse section; b) the thermal conductivity of the heat carrier is zero; c) the temperature of the conductor material is uniform over a transverse section; and d) energy enters and leaves the temperature integrator only with the stream of heat carrier.

On the basis of these assumptions, the dynamics of the thermal processes in a temperature integrator can be described by two differential equations:

$$-c\gamma v\frac{\partial T}{\partial x}dxdt=\frac{C_f}{b}\frac{\partial T}{\partial t}dxdt+\alpha l(T-U)dxdt;$$

$$kml\frac{\partial^2 U}{\partial x^2}dxdt+\alpha l(T-U)dxdt=\frac{C_m}{b}\frac{\partial U}{\partial t}dxdt.$$

We introduce the quantities

$$\beta=\frac{\alpha bl}{c\gamma v},\qquad a^2=\frac{kmbl}{C_m},\qquad \tau_f=\frac{C_f}{c\gamma v},\qquad \tau_m=\frac{C_m}{c\gamma v}.$$

The differential equations can then be transformed into:

$$b\frac{\partial T}{\partial x}+\tau_f\frac{\partial T}{\partial t}+\beta(T-U)=0, \tag{1}$$

$$\frac{\partial U}{\partial t}=a^2\frac{\partial^2 U}{\partial x^2}+\frac{\beta}{\tau_m}(T-U). \tag{2}$$

The solution to our problem will be sought for zero initial and boundary conditions which follows from the preceding assumptions:

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$$T(x,\ 0)=0, \qquad U(x,\ 0)=0$$

and

$$\frac{\partial U(0,\ t)}{\partial x}=0, \qquad \frac{\partial U(b,\ t)}{\partial x}=0,$$

$$T(0,\ t)=T_0(t), \qquad T(b,\ t)=T_e(t).$$

For a solution to the problem, we apply the Laplace transformation to (1) and (2). Subsequent algebraic operations in (1) and (2) yield

$$\frac{dT_L(x,\ p)}{dx}+\frac{p\tau_f+\beta}{b}T_L(x,\ p)=\frac{\beta}{b}U_L(x,\ p), \tag{3}$$

$$T_L(x,\ p)=\frac{1}{\beta}\left[(p\tau_m+\beta)U_L(x,\ p)-a^2\tau_m\frac{d^2U_L(x,\ p)}{dx^2}\right]. \tag{4}$$

With the aid of (3), we express the transform of function $T_e(t)$ in terms of the transforms of functions $U(x,\ t)$ and $T_0(t)$ [2]:

$$T_e(p)=T_0(p)\exp(-\delta b)+\frac{\beta}{b}\exp(-\delta b)\int_0^b U_L(x,\ p)\exp(\delta x)\,dx, \tag{5}$$

where

$$\delta=\frac{p\tau_f+\beta}{b}.$$

The temperature distribution $U(x,\ p)$ will be found from the equation

$$\frac{d^3U_L(x,\ p)}{dx^3}+\frac{p\tau_f+\beta}{b}\frac{d^2U_L(x,\ p)}{dx^2}-\frac{p\tau_m+\beta}{a^2\tau_m}\frac{dU_L(x,\ p)}{dx}$$
$$+\frac{[\beta-(p\tau_m+\beta)(p\tau_f+\beta)]}{a^2\tau_m b}U_L(x,\ p)=0, \tag{6}$$

which is obtained by eliminating $T_L(x,\ p)$ from system (3)-(4).

Solving Eq. (6) with the boundary conditions

$$\frac{\partial U(0,\ t)}{\partial x}=\frac{\partial U(b,\ t)}{\partial x}=0,$$

which correspond to a thermal insulation of the TI end surfaces, then considering the relation

$$\int_0^b U_L(x,\ p)\,dx=\frac{\beta b\,[T_e(p)-T_0(p)]}{\beta^2-(p\tau_f+\beta)(p\tau_m+\beta)},$$

obtained from (3)-(4) and expressing the Law of Energy Conservation applied to a temperature integrator, and finally inserting the result into (5), we find

$$T_e(p)=T_0(p)\exp(-\delta b)+[T_e(p)-T_0(p)]\,M(p), \tag{7}$$

where

$$M(p)=\frac{\beta^2}{\beta^2-(p\tau_m+\beta)\ (p\tau_f+\beta)}N(p),$$

$$N(p)=\left\{\frac{r_2r_3}{r_1+\delta}[\exp(r_3b)-\exp(r_2b)]\ [\exp(r_1b)-\exp(-\delta b)]\right.$$
$$+\frac{r_1r_3}{r_2+\delta}[\exp(r_1b)-\exp(r_3b)]\ [\exp(r_2b)-\exp(-\delta b)]$$

$$+\frac{r_1r_2}{r_3+\delta}[\exp(r_2b)-\exp(r_1b)][\exp(r_3b)-\exp(-\delta b)]\Big\}\Big\{\frac{r_2r_3}{r_1}[\exp(r_3b)-\exp(r_2b)][\exp(r_1b)-1]$$

$$+\frac{r_1r_3}{r_2}[\exp(r_1b)-\exp(r_3b)][\exp(r_2b)-1]+\frac{r_1r_2}{r_3}[\exp(r_2b)-\exp(r_1b)][\exp(r_3b)-1]\Big\}^{-1}, \tag{7a}$$

where r_1, r_2, and r_3 are the roots of the characteristic equation

$$y^3+\frac{p\tau_f+\beta}{b}y^2-\frac{p\tau_m+\beta}{a^2\tau_m}y-\frac{p^2\tau_m\tau_f+p\beta(\tau_m+\tau_f)}{a^2\tau_m b}=0.$$

From (7) we find the transient characteristic of a temperature integrator in the form

$$K(p)=\frac{T_e(p)}{T_0(p)}=\exp(-\delta b)-\frac{\beta^2[1-\exp(-\delta b)]N(p)}{(p\tau_m+\beta)(p\tau_f+\beta)-\beta^2+\beta^2N(p)}. \tag{8}$$

Letting $p = i\omega$ in (8), according to the standard procedure, we obtain an expression for the frequency characteristic of a temperature integrator:

$$K(i\omega)=\exp(-\beta-i\omega\tau_f)-\frac{\beta^2[1-\exp(-\beta-i\omega\tau_f)]N(i\omega)}{-\omega^2\tau_m\tau_f+i\omega\beta(\tau_m+\tau_f)+\beta^2N(i\omega)} \tag{9}$$

and an equation for finding the roots r_1, r_2, r_3 in $N(i\omega)$:

$$y^3+\left(\frac{\beta}{b}+i\frac{\omega\tau_f}{b}\right)y-\left(\frac{\beta}{a^2\tau_m}+\frac{i\omega}{a^2}\right)y+\frac{\omega^2\tau_m}{a^2b}-i\frac{\omega\beta(\tau_f+\tau_m)}{a^2\tau_m b}=0. \tag{10}$$

It follows from (9) that the maximum factor of noise attenuation at $\omega = \infty$ will be $\exp(-\beta)$, because the second term in (9) then approaches zero.

Parameter β, which characterizes the maximum attenuation of noise in a temperature integrator, may also be called the interaction parameter, as it determines the rate of heat transfer from the heat carrying fluid to the TI material.

We will now consider a few special cases. Up to frequencies determined from the inequality

$$1\geqslant|K(i\omega)|>\exp(-\beta), \tag{11}$$

the amplitude-frequency characteristic $K(\omega)$ and the phase-frequency characteristic $\Delta\varphi(\omega)$ of a temperature integrator can, according to (9), be expressed as

$$K(\omega)=\beta^2\sqrt{\frac{(N_1^2+N_2^2)[1-2\exp(-\beta)\cos\omega\tau_f+\exp(-2\beta)]}{[\beta N_1-\omega^2\tau_f\tau_m]^2+[\omega\beta(\tau_m+\tau_f)+\beta N_2]^2}}, \tag{12}$$

$$\Delta\varphi(\omega)=(-1)^{\frac{1}{2}\left(1-\frac{A_1}{|A_1|}\right)}\operatorname{arctg}\frac{A_2}{|A_1|}-(-1)^{\frac{1}{2}\left(1-\frac{B_1}{|B_1|}\right)}\operatorname{arctg}\frac{B_2}{|B_1|}+\frac{\pi}{2}\left(\frac{B_1}{|B_1|}-\frac{A_1}{|A_1|}\right), \tag{13}$$

where N_1 and N_2 are, respectively, the real and the imaginary part of $N(i\omega)$,

$$A_1=N_1[1-\exp(-\beta)\cos\omega\tau_f]-N_2\exp(-\beta)\sin\omega\tau_f;$$
$$A_2=N_1\exp(-\beta)\sin\omega\tau_f+N_2[1-\exp(-\beta)\cos\omega\tau_f];$$
$$B_1=\beta^2N_1-\omega^2\tau_f\tau_m;$$
$$B_2=\omega\beta(\tau_f+\tau_m)+\beta^2N_2.$$

If $a^2 = \infty$ or $b = \infty$, then

$$N(i\omega)=\frac{1-\exp(-\beta-i\omega\tau_f)}{\beta+i\omega\tau_f}, \tag{14}$$

and instead of (12)-(13) we have, respectively, for $\exp(-\beta) \ll 1$

$$K(\omega)=\left\{\left[1-\omega^2\frac{\tau_f}{\beta}(\tau_f+2\tau_m)\right]^2+\left[\omega(\tau_m+\tau_f)-\omega^3\left(\frac{\tau_f}{\beta}\right)^2\tau_m\right]^2\right\}^{-\frac{1}{2}}, \tag{15}$$

$$\Delta\varphi(\omega) = (-1)^{\frac{1}{2}\left(1-\frac{C_1}{|C_1|}\right)} \operatorname{arctg} \frac{C_2}{|C_1|} - \frac{\pi}{2}\left(1 - \frac{C_1}{|C_1|}\right), \quad (16)$$

where

$$C_1 = 1 - \omega^2 \frac{\tau_f}{\beta}(\tau_f + 2\tau_m);$$

$$C_2 = \omega^3 \left(\frac{\tau_f}{\beta}\right)^2 \tau_m - \omega(\tau_m + \tau_f).$$

When $\beta = \infty$, (15) becomes

$$K(\omega) = \frac{1}{\sqrt{1 + \omega^2(\tau_m + \tau_f)^2}}, \quad (17)$$

which corresponds to an infinitely high thermal diffusivity of the TI material [1].

A comparison will show that, when $\beta > 1/3$, expression (15) diverges from the simple expression (17) for an ideal inertia element at frequencies ω

$$\omega > \omega_0 = \sqrt{\frac{\beta}{\tau_f(\tau_f + 2\tau_m)}}.$$

We note that, as an approximation, the frequency characteristics (15) and (16) can be used also in the following situations.

1. Situations of the kind where $a^2 = \infty$, when the roots of Eq. (10) satisfy the inequalities $|r_1| \gg |r_2|$ and $|r_1| \gg |r_3|$, will occur, if

$$k_1 = \frac{a^2\tau_m}{b} \cdot \frac{\beta^2 + \omega^2\tau_f^2}{\beta^2 + \omega^2\tau_m^2},$$

$$k_2 = \frac{a^2\tau_m}{b} \cdot \frac{\beta^2 + \omega^2\tau_f^2}{\omega^4\tau_f^2\tau_m^2 + \omega^2\beta\tau_m(\tau_m + \tau_f)}$$

satisfy the inequalities $k_1 > 1$ and $k_2 > 1$. Quantities k_1 and k_2 define the convergence of (15) and (16) to (12) and (13), respectively. If $k_1 = k_2 = \infty$, then (15) and (16) will be the frequency characteristics.

2. Situations of the kind where $b = \infty$, when the roots $r_1 \approx -r_2$ and r_3 satisfy the inequalities $|r_1| \gg |r_3|$, $|r_2| \gg |r_3|$,

$$\left|1 - \left|\frac{r_1}{r_2}\right|\right| \approx \frac{a\sqrt{\tau_m}}{b}\left|\frac{\beta + i\omega\tau_f}{\sqrt{-\beta - i\omega\tau_m}}\right| \ll 1,$$

will occur, if quantities l_j (j = 1, 2, 3, 4, 5) defined as

$$l_1 = \frac{\sqrt{\beta^2 + \omega^2\tau_m^2}}{\omega^2\tau_m^2\left[\omega^2\tau_f^2 + \beta^2\left(1 + \frac{\tau_f}{\tau_m}\right)^2\right]},$$

$$l_2 = \frac{b^2}{\omega^2 a^2\tau_m^3} \cdot \frac{(\beta + \omega^2\tau_m^2)^{\frac{3}{2}}}{\left[\omega^2\tau_f^2 + \beta^2\left(1 + \frac{\tau_f}{\tau_m}\right)^2\right]},$$

$$l_3 = \frac{b^2}{a^2\tau_m} \cdot \frac{\beta^2 + \omega^2\tau_m^2}{\beta^2 + \omega^2\tau_f^2},$$

$$l_4 = \frac{b^2(\beta^2 + \omega^2\tau_m^2)\left[1 - \exp\left(-\frac{b}{a\sqrt{2\tau_m}}\sqrt{\sqrt{\beta^2 + \omega^2\tau_m^2} - \beta}\right)\right]}{a^2\tau_m^2\omega\left[1 + \exp(-\beta)\right]\sqrt{\beta^2 + \omega^2\tau_f^2}\sqrt{\omega^2\tau_f^2 + \beta^2\left(1 + \frac{\tau_f}{\tau_m}\right)^2}},$$

$$l_5 = \frac{b^3\,(\beta^2+\omega^2\tau_m^2)^{\frac{7}{4}}\,\mathrm{th}\left(\frac{b}{2a\sqrt{2\tau_m}}\sqrt{\sqrt{\beta^2+\omega^2\tau_m^2}-\beta}\right)}{a^3\tau_m^{\frac{5}{2}}\omega\,(\beta^2+\omega^2\tau_f^2)\sqrt{\omega^2\tau_f^2+\beta^2\left(1+\frac{\tau_f}{\tau_m}\right)^2}},$$

satisfy the inequalities $l_j > 1$. Quantities l_j define the convergence of (15) and (16) to (12) and (13), respectively. If $l_j = \infty$, then the frequency characteristics (12) and (13) become identical to (15) and (16), respectively.

The expressions for k_1, k_2, and l_j take into account the given relations between the roots represented in terms of TI parameters and frequency ω; they also take into account that replacing (7a) by (14) results in an error which is a function of k_1, k_2, l_j but becomes zero when $k_1 = k_2 = l_j = \infty$.

The amplitude-frequency and the phase-frequency characteristics of a temperature integrator with $a^2 = 0$ become

$$K(\omega) = \exp\left(-\frac{\omega^2\tau_m^2\beta}{\beta^2+\omega^2\tau_m^2}\right), \tag{18}$$

$$\Delta\varphi(\omega) = -\omega\left(\tau_f + \frac{\tau_m\beta^2}{\beta^2+\omega^2\tau_m^2}\right) \tag{19}$$

and are more conveniently found directly from system (3)-(4) with $a^2 = 0$ in (4).

For a verification of the frequency characteristics (15) and (16), curves have been plotted for a temperature integrator (test accuracy within 20%) which is used in a circular instrument for measuring the radiation power of optical quantum generators.

This temperature integrator was made of aluminum into a rectangular paralellepiped, with barriers installed inside along the path of the fluid stream so as to lengthen the path by zigzagging and thus to maximize the surface area of thermal interaction between the heat carrying fluid and the TI material. In order to increase the conductance of the TI volume, these barriers were joined by rods running parallel to the inlet–outlet direction. This brought the characteristics of the temperature integrator closer to the characteristics of the most effective ideal temperature integrator (17). The time constants of the temperature integrator were in this case $\tau_f = 500$ sec and $\tau_m = 440$ sec.

In Fig. 1 are shown the theoretical and the experimental frequency characteristics of the tested device. Curve 1 is the measured amplitude-frequency characteristic and curve 3 is the measured phase-frequency characteristic of this temperature integrator. The theoretical amplitude-frequency characteristic (curve 2) and phase-frequency characteristic (curve 4) have been calculated by formulas (15) and (16), respectively. They have been plotted with the effective value β_E, which takes into account, to the first approximation, the differences between the TI model (15)-(16) and the test TI:

$$\beta_E = \omega_{90}^2\tau_f\,(\tau_f + 2\tau_m), \tag{20}$$

where ω_{90} is the measured frequency at which the phase characteristic crosses 90°.

Formula (20) for β_E has been derived from the condition that a phase shift of 90° occurs at frequency ω_{90} on both the measured and the calculated characteristic ("tie-in" at one point).

Thus, with the aid of these theoretical characteristics, one can adequately well explain the trend of the measured temperature integrator characteristic.

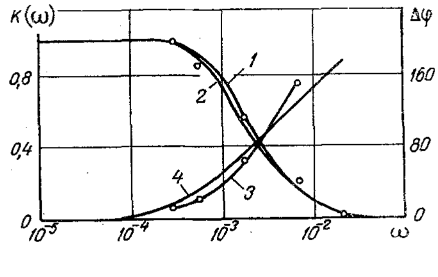

Fig. 1. Frequency characteristics of the temperature integrator: φ (°C); ω (rad/sec).

We note that these results yield an analytical expression for the temperature distribution along a thin rectangular strip of heat conducting material and also in the stream, when the temperature at the inlet to the device varies arbitrarily.

NOTATION

c	is the specific heat of heat carrying fluid;
v	is the volume flow rate of heat carrying fluid;
γ	is the density of heat carrying fluid;
C_m	is the heat capacity of heat conducting material of temperature integrator;
C_f	is the heat capacity of heat carrying fluid inside the TI volume;
α	is the coefficient of heat transfer at conductor-carrier interface;
a^2	is the thermal diffusivity of heat conducting material;
k	is the thermal conductivity of heat conducting material;
b, l, m	are the length, width, and thickness of temperature integrator;
β	is the interaction parameter;
p	is the complex variable;
$T(x, t)$	is the temperature of heat carrier as a function of time and the length coordinate in the direction of flow;
$U(x, t)$	is the temperature of the heat conductor as a function of time and the length coordinate in the direction of flow;
$T_0(t)$	is the temperature at the TI inlet as a function of time;
$T_e(t)$	is the temperature at the TI outlet as a function of time;
$T_L(x, p)$	is the Laplace transform of function $T(x, t)$;
$U_L(x, p)$	is the Laplace transform of function $U(x, t)$;
$T_0(p)$	is the Laplace transform of function $T_0(t)$;
$T_e(p)$	is the Laplace transform of function $T_e(t)$.

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